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HOMEOMORPHISMS OF HILBERT CUBE MANIFOLDS

BY

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ABSTRACT. It is shown in this paper that the homeomorphism group of any compact Hilbert cube manifold is locally contractible. The proof uses some standard infinite-dimensional techniques along with an infinite-dimensional version of the torus-homeomorphism idea which was used by Edwards and Kirby to establish a corresponding finite-dimensional result.

1. Introduction. Let the Hilbert cube be denoted by Q and define a *Hilbert cube manifold* (or Q-manifold) to be a separable metric space which has an open cover by sets which are homeomorphic to open subsets of Q. Some obvious examples of Q-manifolds are (1) open subsets of Q and (2) $M \times Q$, for any finite-dimensional manifold M. Some nonobvious examples of Q-manifolds are provided by the work of West [14], where it is shown that $|K| \times Q$ is a Q-manifold, for any finite complex K.

Recent results by the author indicate that Q-manifolds might play a wider role in topology than was first thought. In [9] it was shown that Borsuk's notion of shape (for compacta) could be characterized in terms of homeomorphic Q-manifolds and in [10] it was shown that Whitehead's notion of simple homotopy equivalences could be characterized in terms of homeomorphisms on Q-manifolds, with this characterization being used to give a proof of the topological invariance of Whitehead torsion. The main idea used in this latter result was an infinite-dimensional version of the torus-homeomorphism technique which was so useful in triangulating finite-dimensional manifolds [13].

In this paper we also use this torus-homeomorphism technique to establish some Q-manifold analogues of the Edwards-Kirby results concerning the local contractibility of the homeomorphism groups of compact n-manifolds [11]. Our main results are the following theorems.

Theorem 1. *The homeomorphism group of any compact Q-manifold is locally contractible.*

Theorem 2. *Let X be a Q-manifold, $U \subset X$ be open, and let $C \subset U$ be compact.*

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If $f_t: U \longrightarrow X$, $t \in I = [0, 1]$, *is an isotopy of open embeddings, then the restrictions* $f_0 | C$ *and* $f_1 | C$ *are ambient isotopic, i.e. there exists an ambient isotopy* $g_t: X \longrightarrow X$, $t \in I$, *such that* $g_0 = \mathrm{id}_X$ *(the identity of X) and* $g_1 \circ f_0 = f_1$ *on C.*

In [15] Wong used a coordinate-switching technique to prove that any homeomorphism on Q is isotopic to the identity, and it follows immediately from the techniques of that paper that the homeomorphism group of Q is contractible and locally contractible. Thus Theorem 1 generalizes this result of Wong. Also it is hoped that Theorem 1 might be useful in investigating the following open question: *Is the homeomorphism group of any compact Q-manifold an ANR?*

For the proofs of Theorems 1 and 2 we will have to use a considerable amount of infinite-dimensional machinery. Some of the ideas used are (1) a version of Wong's coordinate-switching technique, (2) Anderson's notion of *Property Z* [1], and (3) the notion of a *pseudo-boundary* of a Q-manifold. In §2 we give the basic definitions and in §3 we summarize precisely what infinite-dimensional results are needed. Also in Lemma 3.3 we isolate the key idea of [11] which we will need.

2. **Definitions and notation.** For any topological space X and $A \subset X$ we use $\mathrm{Int}_X A$ to denote the topological interior of A in X and $\mathrm{Bd}_X A$ to denote the topological boundary of A in X. When there is no ambiguity we will omit the subscripts. For any finite-dimensional manifold M we use ∂M for the boundary of M.

Let E^n denote Euclidean n-space and let $B^n = [-1, 1]^n \subset E^n$. In general let $B^n_r = [-r, r]^n$, for $r > 0$. Let $e: E^1 \longrightarrow S^1$ denote the covering projection given by $e(x) = \exp(\frac{1}{4}\pi i x)$. Let $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times) be the n-torus and let $e^n = e \times e \times \cdots \times e: E^n \longrightarrow T^n$ be the product covering projection.

We will use the representation $Q = \prod_{i=1}^\infty I_i$, where each I_i is the closed interval $[-1, 1]$, and we let $s = \prod_{i=1}^\infty I_i^0$ where each I_i^0 is the open interval $(-1, 1)$. The standard *pseudo-boundary* of Q is the set $B(Q) = Q \backslash s$. If X is any Q-manifold, then X is homeomorphic to $X \times Q$ [2]. A set $B \subset X$ is a *pseudo-boundary* of X provided that there exists a homeomorphism of X onto $X \times Q$ which takes B onto $X \times B(Q)$. If $U \subset X$, then a function $f: U \longrightarrow X$ is said to be *B-proper* provided that $f^{-1}(B) = U \cap B$. In general a homeomorphism of X onto itself does not have to be B-proper, and in many cases we will have to adjust the homeomorphism so that it is B-proper. For technical reasons of this sort we will use many results from [6] concerning pseudo-boundaries of Q-manifolds.

We will also need the notion of a *Z-set*, introduced by Anderson in [1]. A closed subset K of a space X is said to be a *Z-set* provided that given any nonnull and homotopically trivial open set $U \subset X$, $U \backslash K$ is also nonnull and homotopically trivial.

By a *map* we will always mean a continuous function and for any function $f: X \longrightarrow Y$ and $A \subset X$, we use $f \mid A$ for the restriction of f to A. All of our homeomorphisms will be onto and when we say *embedding* we mean a homeomorphism onto its image. For any space X we use id_X to denote the identity of X and for $A \subset X$ we will (incorrectly) use id_A to denote the inclusion of A in X. We will omit the subscript when the meaning is clear. An *isotopy* of a space X into a space Y is a map $\phi: X \times I \longrightarrow Y$ such that each *level* $\phi_t: X \longrightarrow Y$ is an embedding, where $\phi_t(x) = \phi(x, t)$. An isotopy is *ambient* provided that each level is onto. We use $g \circ f$ for the composition of maps and for isotopies $\phi: X \times I \longrightarrow X$, $\psi: X \times I \longrightarrow X$, with $\psi_0 = \mathrm{id}_X$, we use $\psi * \phi: X \times I \longrightarrow X$ to denote the isotopy defined by

$$\psi * \phi(x, t) = \begin{cases} \phi(x, 2t) & \text{for } 0 \le t \le \tfrac{1}{2}, \\ \psi(\phi_1(x), 2t - 1) & \text{for } \tfrac{1}{2} \le t \le 1. \end{cases}$$

We call $\psi * \phi$ the *composition* of ψ and ϕ.

All function spaces will be equipped with the compact-open topology, and when we say that two functions are *close*, we mean that they are close in the compact-open topology. For any space X and $K \subset X$ we let $H(X, K)$ denote the space of all homeomorphisms $h: X \longrightarrow X$ which satisfy $h \mid K = \mathrm{id}$. If X is a Q-manifold and B is a pseudo-boundary of X, then $H_B^*(X, K)$ will denote the space of all homeomorphisms $h: X \longrightarrow X$ which satisfy $h \mid K = \mathrm{id}$ and $h(B) = B$. When there is no ambiguity we will omit the subscript. We remark that if X is any Q-manifold, then $H(X)$ is a topological group [3].

3. **Some basic lemmas.** In this section we describe some results concerning homeomorphism groups of Q-manifolds which will be used in the sequel. The first result establishes the contractibility of certain homeomorphism groups. It is used in the proof of Lemma 4.1.

Lemma 3.1. *If $K \subset Q$ is a Z-set and B is a pseudo-boundary of Q, then $H_B^*(Q, K)$ contracts to id_Q. Moreover id_Q is fixed at each level of the contraction.*

Proof. W. Barit has shown that $H(Q, K)$ contracts to id_Q (see Theorem 7.11 (2) of [4]). The technique used there applies equally well to prove that $H_{B(Q)}^*(Q, K)$ also contracts to id_Q. There exists a homeomorphism of Q onto itself which takes B onto $B(Q)$. Thus there exists a map

$$\phi: H_B^*(Q, K) \times I \longrightarrow H_B^*(Q, K)$$

which satisfies $\phi_0(h) = h$ and $\phi_1(h) = \mathrm{id}_Q$, for all $h \in H_B^*(Q, K)$. Then define

$$\psi: H_B^*(Q, K) \times I \longrightarrow H_B^*(Q, K)$$

by setting $\psi_t(h) = [\phi_t(\mathrm{id}_Q)]^{-1} \circ \phi_t(h)$, for all $h \in H_B^*(Q, K)$ and $t \in I$. Note that $\psi_0(h) = h$, $\psi_1(h) = \mathrm{id}_Q$, and $\psi_t(\mathrm{id}_Q) = \mathrm{id}_Q$, for all h and t. Moreover it follows that ψ is continuous, since $H_B^*(Q, K)$ is a topological group.

The following result is also used in the proof of Lemma 4.1. It is a canonical homeomorphism extension theorem for Z-sets in Q-manifolds.

Lemma 3.2. *Let X be a Q-manifold, B be a pseudo-boundary of X, and let $H, K \subset X$ be compact Z-sets such that $H \subset K$. If $h: K \to X$ is a B-proper embedding such that $h(K)$ is a Z-set and $h \mid H = \mathrm{id}$, then h can be chosen sufficiently close to id_K so that there exists an ambient isotopy $\phi: X \times I \to X$ such that*

(1) *ϕ depends continuously on h,*

(2) *$\phi_0 = \mathrm{id}$ and $\phi_1 \circ h = \mathrm{id}_K$,*

(3) *$\phi_t = \mathrm{id}$ for all t if $h = \mathrm{id}$,*

(4) *$\phi_t(B) = B$ and $\phi_t \mid H = \mathrm{id}$, for all t.*

Proof. It follows from Theorem 5.1 (2) of [8] that h can be chosen sufficiently close to id so that an ambient isotopy $\phi: X \times I \to X$ exists which satisfies (1), (2) and the first part of (4). We merely remark that one can construct ϕ to additionally satisfy (3) and the last part of (4) by making some slight adjustments on the proof given in [8].

We will need one more result in the proof of Lemma 4.1. It is essentially Lemma 8.1 of [11], and in fact it can be proven by multiplying everything by X which is used in the proof given there. For this reason we omit the proof.

Lemma 3.3. *Let X be a compact connected metric space, K and B be subsets of X, and let $h: X \times B_4^n \to X \times E^n$ be an embedding such that $h(X \times \mathrm{Int}\, B_4^n)$ is open in $X \times E^n$, $h \mid K \times B_4^n = \mathrm{id}$, and $h^{-1}(B \times E^n) = B \times B_4^n$. Then h can be chosen sufficiently close to id so that there exists a homeomorphism $\hat{h}: X \times T^n \to X \times T^n$ which satisfies the following properties.*

(1) *$\hat{h}$ depends continuously on h,*

(2) *$\hat{h} \mid X \times T^n = \mathrm{id}$ and $\hat{h}(B \times T^n) = B \times T^n$,*

(3) *$\hat{h} = \mathrm{id}$ if $h = \mathrm{id}$,*

(4) *the following diagram commutes:*

$$\begin{array}{ccc} X \times T^n & \xrightarrow{\hat{h}} & X \times T^n \\ {\scriptstyle \mathrm{id} \times e^n} \big\uparrow & & \big\uparrow {\scriptstyle \mathrm{id} \times e^n} \\ X \times B_2^n & \xrightarrow{h} & X \times E^n \end{array}$$

We will need the following result in the proof of Theorem 1.

Lemma 3.4. *Let X be a Q-manifold and let B be a pseudo-boundary of X. Then there exists a map $\phi: H(X) \times I \to H(X)$ such that $\phi_0(h) = h$ and $\phi_t(h) \in H_B^*(X)$, for all $h \in H(X)$ and $t \in (0, 1]$, and $\phi_t(\mathrm{id}) = \mathrm{id}$ for all t.*

Proof. It follows from [6] that there exists a homeomorphism of X onto $X \times Q$ which takes B onto $B_1 = X \times Q_1 \times \sigma$, where $Q_1 = \Pi\{I_i \mid i \text{ odd}\}$ and

$$\sigma = \left\{(x_i) \in \prod\{I_i \mid i \text{ even}\} \mid x_i = 0 \text{ for } i \text{ sufficiently large}\right\}.$$

Thus all we need to do is construct a map ψ: $H(X \times Q) \times I \longrightarrow H(X \times Q)$ which satisfies $\psi_0(h) = h$ and $\psi_t(h) \in H^*_{B_1}(X \times Q)$, for all $h \in H(X \times Q)$ and $t \in (0, 1]$, and $\psi_t(\text{id}) = \text{id}$ for all t. But this is essentially Lemma 7.1 of [8].

We will need the following result in the proof of Theorem 2.

Lemma 3.5. *Let X be a Q-manifold, B be a pseudo-boundary of X, $U \subset X$ be open, and let h: $U \longrightarrow X$ be an open embedding. Then there exists an ambient isotopy ϕ: $X \times I \longrightarrow X$ such that $\phi_0 = \text{id}$ and $\phi_1 \circ h$ is B-proper. Moreover we can construct ϕ so that $\phi_1 \circ h$ is as close to h as we wish.*

Proof. It is easily seen that $h(U \cap B)$ and $h(U) \cap B$ are pseudo-boundaries of $h(U)$ and therefore there exists an ambient isotopy ϕ: $X \times I \longrightarrow X$ such that $\phi_0 = \text{id}$, $\phi_1(h(U \cap B)) = h(U) \cap B$, and $\phi_t \mid X \backslash h(U) = \text{id}$ for all t (see [6]). Moreover ϕ can be constructed so that $\phi_1 \circ h$ is arbitrarily close to h. It is obvious that $\phi_1 \circ h$ is B-proper.

4. **A special case.** The following result is an essential step in the proof of Theorem 5.1. It is an infinite-dimensional analogue of Lemma 4.1 of [11].

Lemma 4.1. *Let $K \subset Q$ be a Z-set and let h: $Q \times B^n_4 \longrightarrow Q \times E^n$ be an embedding such that $h(Q \times \text{Int } B^n_4)$ is open, $h \mid K \times B^n_4 = \text{id}$, and $h^{-1}(B(Q) \times E^n) = B(Q) \times B^n_4$. If h is sufficiently close to* id, *then there is an ambient isotopy ϕ: $Q \times E^n \times I \longrightarrow Q \times E^n$ such that the following properties are satisfied*:

(1) *ϕ depends continuously on h,*

(2) $\phi_0 = \text{id}$ *and* $\phi_1 \circ h \mid Q \times B^n = \text{id}$,

(3) $\phi_t = \text{id}$ *for all t if $h = \text{id}$, and*

(4) $\phi_t(B(Q) \times E^n) = B(Q) \times E^n$ *and* $\phi_t \mid (K \times E^n) \cup (Q \times (E^n \backslash \text{Int } B^n_3)) = \text{id}$ *for all t.*

Proof. For h sufficiently close to id we will construct a homeomorphism g: $Q \times B^n_3 \longrightarrow Q \times B^n_3$ such that (1) g depends continuously on h, (2) $g \mid Q \times B^n = h \mid Q \times B^n$, (3) $g = \text{id}$ if $h = \text{id}$, and (4) $g(B(Q) \times B^n_3) = B(Q) \times B^n_3$ and $g \mid (K \times B^n_3) \cup (Q \times \text{Bd } B^n_3) = \text{id}$. Our required isotopy ϕ will then be defined by appealing to Lemma 3.1. In the accompanying diagram we have indicated the maps involved in the construction of g.

Using Lemma 3.3 we can choose h close enough to id so that there exists a homeomorphism h_1: $Q \times T^n \longrightarrow Q \times T^n$ such that (1) h_1 depends continuously on h,

(2) $h_1 \mid K \times T^n = \mathrm{id}$ and $h_1(B(Q) \times T^n) = B(Q) \times T^n$, (3) $h_1 = \mathrm{id}$ if $h = \mathrm{id}$, and (4) the lower rectangle in the accompaning diagram commutes.

Standard covering space theory implies the h_1 can be lifted to a homeomorphism $h_2: Q \times E^n \to Q \times E^n$. If h_1 is sufficiently close to id, then we can construct h_2 to depend continuously on h_1. To see this let $U_\delta(x)$ denote the open δ-neighborhood of x in E^n and note that $e^n \mid U_3(x)$ is an embedding for all x in E^n. If h_1 is sufficiently close to id, then

$$h_1(Q \times e^n(U_2(x))) \subset Q \times e^n(U_3(x)),$$

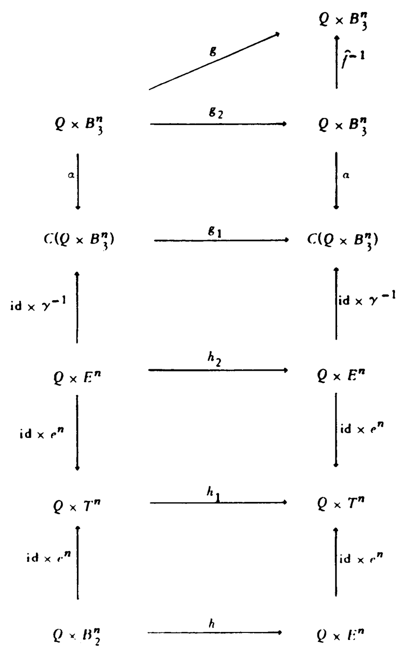
for all $x \in E^n$. Then we define $h_2: Q \times E^n \to Q \times E^n$ by requiring

$$h_2 \mid Q \times U_2(x) = [(\mathrm{id} \times e^n) \mid (Q \times U_3(x))]^{-1} \circ h_1 \circ (\mathrm{id} \times e^n) \mid Q \times U_2(x),$$

for all $x \in E^n$. We note that h_2 is a homeomorphism such that (1) h_2 depends continuously on h_1, (2) $h_2 \mid K \times E^n = \mathrm{id}$ and $h_2(B(Q) \times E^n) = B(Q) \times E^n$, (3) $h_2 = \mathrm{id}$ if $h_1 = \mathrm{id}$, and (4) the second rectangle in the accompanying diagram commutes. From the commutativity of the first and second rectangles we get $h_2 \mid Q \times B_2^n = h \mid Q \times B_2^n$. Let d_1 be any metric for Q and define a metric d on $Q \times E^n$ by

$$d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \|y_1 - y_2\|.$$

Then it follows that h_2 is *bounded*, that is the set $\{d((x, y), h_2(x, y)) \mid (x, y) \in Q \times E^n\}$ is bounded above. This will be needed for the construction of g_1.



Now let $C(Q \times B_3^n) = (Q \times \text{Int } B_3^n) \cup \text{Bd } B_3^n$ and define $p: Q \times B_3^n \longrightarrow C(Q \times B_3^n)$ by

$$p(x, y) = \begin{cases} (x, y) & \text{for } y \in \text{Int } B_3^n, \\ y & \text{for } y \in \text{Bd } B_3^n. \end{cases}$$

Give $C(Q \times B_3^n)$ the identification topology determined by p and let γ: Int $B_3^n \longrightarrow E^n$ be a radial expansion which satisfies $\gamma | B_2^n = \text{id}$. If h_2 is sufficiently close to id we have $h_2(Q \times B_{1.5}^n) \subset Q \times B_2^n$. Then define $g_1: C(Q \times B_3^n) \longrightarrow C(Q \times B_3^n)$ by

$$g_1 = \begin{cases} \text{id} & \text{on Bd } B_3^n, \\ (\text{id} \times \gamma)^{-1} \circ h_2 \circ (\text{id} \times \gamma) & \text{on } Q \times \text{Int } B_3^n. \end{cases}$$

Since h_2 is bounded it follows that g_1 is continuous, therefore a homeomorphism. It also follows that (1) g_1 depends continuously on h_2, (2) $g_1 | K \times \text{Int } B_3^n = \text{id}$ and $g_1(B(Q) \times \text{Int } B_3^n) = B(Q) \times \text{Int } B_3^n$, (3) $g_1 = \text{id}$ if $h_2 = \text{id}$, (4) $g_1 | Q \times B_{1.5}^n = h | Q \times B_{1.5}^n$, and (5) the third rectangle in the accompanying diagram commutes. This completes the construction of g_1.

We now need a homeomorphism $\alpha: Q \times B_3^n \longrightarrow C(Q \times B_3^n)$ which satisfies $\alpha | Q \times B_2^n = \text{id}$ and $\alpha(B(Q) \times \text{Int } B_3^n) = B(Q) \times \text{Int } B_3^n$. Let $C(Q \times [-3, 3]) = (Q \times (-3, 3)) \cup \{-3, 3\}$ and define $q: Q \times [-3, 3] \longrightarrow C(Q \times [-3, 3])$ by $q(x, y) = (x, y)$, for $y \in (-3, 3)$, and $q(x, y) = y$, for $y = -3, 3$. Give $C(Q \times [-3, 3])$ the identification topology determined by q. There are standard techniques for finding a homeomorphism $\beta: Q \times [-3, 3] \longrightarrow C(Q \times [-3, 3])$ which satisfies $\beta | Q \times [-2, 2] = \text{id}$. One way to see this is to regard $q(Q \times [2, 3])$ as the cone over Q and $q(Q \times [-3, -2])$ as the cone over Q, and then apply the well-known theorem that Q is homeomorphic to its own cone. The necessary details can be dug out of [1]. Now let $L \subset B_3^n$ be the intersection of a line through the origin with B_3^n. Then L meets Bd B_3^n in exactly two points. Applying the comments made above to each such $Q \times L$ we can construct a homeomorphism $\alpha_1: Q \times B_3^n \longrightarrow C(Q \times B_3^n)$ so that $\alpha_1 | Q \times B_2^n = \text{id}$.

The next step is to obtain a homeomorphism $\alpha_2: C(Q \times B_3^n) \longrightarrow C(Q \times B_3^n)$ which modifies α_1 so that $\alpha_2 | Q \times B_2^n = \text{id}$ and $\alpha_2 \circ \alpha_1(B(Q) \times \text{Int } B_3^n) = B(Q) \times \text{Int } B_3^n$. If α_2 is so constructed, then $\alpha = \alpha_2 \circ \alpha_1$ will fulfill our requirements. To construct α_2 we note that $X_1 = Q \times (B_3^n \backslash \text{Int } B_2^n)$ and $X_2 = C(Q \times B_3^n) \backslash (Q \times \text{Int } B_2^n)$ are compact Q-manifolds and $\alpha_1(X_1) = X_2$. Moreover we see that $B = B(Q) \times (\text{Int } B_3^n \backslash \text{Int } B_2^n)$ is a pseudo-boundary of X_1 and of X_2. Thus $\alpha_1(B)$ is a pseudo-boundary of X_2 and we can use [6] to find a homeomorphism $\tilde{\alpha}_2: X_2 \longrightarrow X_2$ which satisfies $\tilde{\alpha}_2 \circ \alpha_1(B) = B$ and $\tilde{\alpha}_2 | Q \times \text{Bd } B_2^n = \text{id}$. Then extend $\tilde{\alpha}_2$ by the identity to a homeomorphism $\alpha_2: C(Q \times B_3^n) \longrightarrow C(Q \times B_3^n)$ to fulfill our requirements.

For g_1 sufficiently close to id we define g_2: $Q \times B_3^n \to Q \times B_3^n$ by $g_2 = \alpha^{-1} \circ g_1 \circ \alpha$ and note that (1) g_2 depends continuously on g_1, (2) $g_2(B(Q) \times \text{Int } B_3^n) = B(Q) \times \text{Int } B_3^n$ and $g_2 | K \times B_2^n = \text{id}$, (3) $g_2 = \text{id}$ if $g_1 = \text{id}$, (4) $g_2 | Q \times B_{1.5}^n = h | Q \times B_{1.5}^n$, and (5) the upper rectangle in the accompanying diagram commutes. This completes the construction of g_2. We note that the condition

$$g_2 | (K \times (B_3^n \backslash \text{Int } B_2^n)) \cup (Q \times \text{Bd } B_3^n) = \text{id}$$

may not necessarily be satisfied. The purpose of g is to remedy this defect.

Put

$$A = (K \times (B_3^n \backslash \text{Int } B_{1.5}^n)) \cup (Q \times \text{Bd } B_{1.5}^n) \cup (Q \times \text{Bd } B_3^n)$$

and note that A is a compact Z-set in the compact Q-manifold $X = Q \times (B_3^n \backslash \text{Int } B_{1.5}^n)$. Define f: $A \to X$ by $f | Q \times \text{Bd } B_{1.5}^n = \text{id}$ and $f | (K \times (B_3^n \backslash \text{Int } B_{1.5}^n)) \cup (Q \times \text{Bd } B_3^n) = g_2 | (K \times (B_3^n \backslash \text{Int } B_{1.5}^n)) \cup (Q \times \text{Bd } B_3^n)$. Note that f depends continuously on g_2, $f = \text{id}$ if $g_2 = \text{id}$, $f(A)$ is a Z-set, and f is an embedding if g_2 is sufficiently close to id.

Now put $B_1 = B(Q) \times (\text{Int } B_3^n \backslash B_{1.5}^n)$, which is a pseudo-boundary of X. Moreover we note that if g_2 is sufficiently close to id, then f is B_1-proper. Using Lemma 3.2 we can choose f sufficiently close to id so that there exists a homeomorphism $\tilde{f}$: $X \to X$ such that (1) $\tilde{f}$ depends continuously on f, (2) $\tilde{f}$ extends f, (3) $\tilde{f} = \text{id}$ if $f = \text{id}$, and (4) $\tilde{f}(B_1) = B_1$. Extend $\tilde{f}$ by the identity to a homeomorphism $\hat{f}$: $Q \times B_3^n \to Q \times B_3^n$ and define g: $Q \times B_3^n \to Q \times B_3^n$ by $g = \hat{f}^{-1} \circ g_2$. We note that for g_2 sufficiently close to id we get $g | Q \times B^n = h | Q \times B^n$.

Let $C = (K \times B_3^n) \cup (Q \times \text{Bd } B_3^n)$, which is a Z-set in $Q \times B_3^n$, and let $B_2 = B(Q) \times B_3^n$, which is a pseudo-boundary of $Q \times B_3^n$. Using Lemma 3.1 there exists an isotopy $\tilde{\phi}$: $Q \times B_3^n \times I \to Q \times B_3^n$ such that (1) $\tilde{\phi}$ depends continuously on g, (2) $\tilde{\phi}_0 = \text{id}$ and $\tilde{\phi}_1 \circ g = \text{id}$, (3) $\tilde{\phi}_t | C = \text{id}$ and $\tilde{\phi}_t(B_2) = B_2$ for all t. Then extend $\tilde{\phi}$ by the identity to an isotopy ϕ: $Q \times E^n \times I \to Q \times E^n$. It is easy to see that ϕ fulfills our requirements.

5. **The main result.** We now state and prove the strongest theorem of this paper. The idea of the proof is to reduce the problem so that repeated applications of Lemma 4.1 can be used.

Theorem 5.1. *Let X be a Q-manifold, B be a pseudo-boundary of X, $U \subset X$ be open, and let $K \subset U$ be compact. If h: $U \to X$ is a B-proper open embedding, then h can be chosen sufficiently close to* id *so that there exists an ambient isotopy ϕ: $X \times I \to X$ which satisfies the following properties.*

(1) *ϕ depends continuously on h,*

(2) *$\phi_0 = \text{id}$ and $\phi_1 \circ h | K = \text{id}$,*

(3) $\phi_t = \mathrm{id}$ *for all* t *if* $h = \mathrm{id}$,

(4) $\phi_t(B) = B$ *and* $\phi_t \mid X \setminus U = \mathrm{id}$ *for all* t.

Proof. The open embedding theorem of [7] implies that X has two coordinate neighborhoods, i.e. there exist open subsets V_1, V_2 of X such that $X = V_1 \cup V_2$ and there exist open embeddings $\alpha_1: V_1 \to Q$, $\alpha_2: V_2 \to Q$. Choose compact sets $K_1, K_2, \tilde{K}_1, \tilde{K}_2$ such that $\tilde{K}_1 \cup \tilde{K}_2 \subset U$, $K = K_1 \cup K_2$, $K_i \subset \mathrm{Int}(\tilde{K}_i) \subset \tilde{K}_i \subset V_i$, for $i = 1, 2$, and put $\tilde{K} = \tilde{K}_1 \cup \tilde{K}_2$. We will construct ambient isotopies $\phi^1: X \times I \to X$ and $\phi^2: X \times I \to X$ such that $\phi = \phi^2 * \phi^1$ fulfills our requirements. We will only give the details for ϕ^2, as the construction of ϕ^1 is just a special case of the construction of ϕ^2. Thus we assume that we have an ambient isotopy ϕ^1: $X \times I \to X$ such that (1) ϕ^1 depends continuously on h, (2) $\phi^1_0 = \mathrm{id}$ and $\phi^1_1 \circ h \mid \tilde{K}_1 = \mathrm{id}$, (3) $\phi^1_t = \mathrm{id}$ for all t if $h = \mathrm{id}$, and (4) $\phi^1_t(B) = B$ and $\phi^1_t \mid X \setminus U = \mathrm{id}$ for all t.

Note that $\alpha_2(U \cap V_2)$ is an open subset of Q which contains $\alpha_2(\tilde{K}_2)$. Choose an integer n such that

$$\alpha_2(\tilde{K}_2) \subset C \times Q_1 \subset G \times Q_1 \subset \alpha_2(U \cap V_2),$$

where $C \subset \Pi_{i=1}^n I_i$ is compact, $G \subset \Pi_{i=1}^n I_i$ is open, and $Q_1 = \Pi_{i=n+1}^\infty I_i$. We can find a compact n-manifold M such that $C \subset \mathrm{Int}_G M \subset M \subset G$ and M is a piecewise linear (PL) subspace of $\Pi_{i=1}^n I_i$. We now use the apparatus of [6] to adjust α_2. From [6] it follows that $B_1 = \alpha_2(B \cap \alpha_2^{-1}(M \times Q_1))$ and $B_2 = M \times B(Q_1)$ are pseudo-boundaries of $M \times Q_1$. Thus there exists a homeomorphism $\sigma: M \times Q_1 \to M \times Q_1$ such that $\sigma \mid \mathrm{Bd}_G M \times Q_1 = \mathrm{id}$ and $\sigma(B_1 \setminus (\mathrm{Bd}\, M \times Q_1)) = B_2 \setminus (\mathrm{Bd}\, M \times Q_1)$. Now put $\tilde{\alpha}_2 = \sigma \circ \alpha_2 \mid \alpha_2^{-1}(M \times Q_1)$, $H = \tilde{\alpha}_2(K \cap \tilde{\alpha}_2^{-1}(M \times Q_1))$, and $\tilde{H} = \tilde{\alpha}_2(\tilde{K} \cap \tilde{\alpha}_2^{-1}(M \times Q_1))$. For h sufficiently close to id we can define an embedding $g: \tilde{H} \to M \times Q_1$ by setting $g = \tilde{\alpha}_2 \circ \phi^1_1 \circ h \circ \tilde{\alpha}_2^{-1} \mid \tilde{H}$. It follows that $g^{-1}(M \times B(Q_1)) = \tilde{H} \cap (M \times B(Q_1))$, $g \mid (\mathrm{Bd}\, M \times Q_1) \cap \tilde{H} = \mathrm{id}$, and $g = \mathrm{id}$ if $h = \mathrm{id}$. Note also that $g(H) \subset \mathrm{Int}\, g(\tilde{H})$. For g sufficiently close to id we will construct an ambient isotopy $\tilde{\phi}: M \times Q_1 \times I \to M \times Q_1$ such that (1) $\tilde{\phi}$ depends continuously on g, (2) $\tilde{\phi}_0 = \mathrm{id}$ and $\tilde{\phi}_1 \circ g \mid H = \mathrm{id}$, (3) $\tilde{\phi}_t = \mathrm{id}$ for all t if $g = \mathrm{id}$, and (4) $\tilde{\phi}_t(M \times B(Q_1)) = M \times B(Q_1)$ and $\tilde{\phi}_t \mid \mathrm{Bd}\, M \times Q_1 = \mathrm{id}$ for all t. Once $\tilde{\phi}$ is constructed we can then define $\phi^2: X \times I \to X$ by setting

$$\phi^2_t = \begin{cases} \mathrm{id} & \text{on } X \setminus \tilde{\alpha}_2^{-1}(M \times Q_1), \\ \tilde{\alpha}_2^{-1} \circ \tilde{\phi}_t \circ \tilde{\alpha}_2 & \text{on } \tilde{\alpha}_2^{-1}(M \times Q_1). \end{cases}$$

Thus the entire problem reduces to the construction of $\tilde{\phi}$. We will construct $\tilde{\phi}$ as the composition of isotopies ψ and θ.

1. *Construction of* ψ. We will construct an ambient isotopy $\psi: M \times Q_1 \times I \to M \times Q_1$ such that (1) ψ depends continuously on g, (2) $\psi_0 = \mathrm{id}$ and $\psi_1 \circ g \mid V = \mathrm{id}$,

where $V \subset M \times Q_1$ is a neighborhood of $(\partial M \times Q_1) \cap H$, (3) $\psi_t = \mathrm{id}$ for all t if $g = \mathrm{id}$, and (4) $\psi_t(M \times B(Q_1)) = M \times B(Q_1)$ and $\psi_t \mid \mathrm{Bd}\, M \times Q_1 = \mathrm{id}$ for all t. We construct ψ as the composition of isotopies ψ^1 and ψ^2.

For the construction of ψ^1 let $A = (\mathrm{Bd}\, M \times Q_1) \cup (\tilde{H} \cap (\partial M \times Q_1))$, which is clearly a Z-set in $M \times Q_1$. Let $f\colon A \longrightarrow M \times Q_1$ be defined by $f \mid \mathrm{Bd}\, M \times Q_1 = \mathrm{id}$ and $f \mid \tilde{H} \cap (\partial M \times Q_1) = g \mid \tilde{H} \cap (\partial M \times Q_1)$. Then $f(A)$ is a Z-set in $M \times Q_1$ and $f(A \cap (M \times B(Q_1)) = f(A) \cap (M \times B(Q_1))$. Using Lemma 3.2 we can choose f sufficiently close to id so that there exists an ambient isotopy $\psi^1\colon M \times Q_1 \times I \longrightarrow M \times Q_1$ such that (1) ψ^1 depends continuously on f, (2) $\psi_0^1 = \mathrm{id}$ and $\psi_1^1 \circ f = \mathrm{id}$, (3) $\psi_t^1 = \mathrm{id}$ for all t if $f = \mathrm{id}$, and (4) $\psi_t^1(M \times B(Q_1)) = M \times B(Q_1)$ and $\psi_t^1 \mid \mathrm{Bd}\, M \times Q_1 = \mathrm{id}$ for all t.

For the construction of ψ^2 we use a modification of the technique of Proposition 3.2 of [11]. To simplify notation we assume that n was chosen large enough so that there exist compact sets $D_1, D_2 \subset M$ such that

$$H \subset D_1 \times Q_1 \subset (\mathrm{Int}\, D_2) \times Q_1 \subset D_2 \times Q_1 \subset \mathrm{Int}\, \tilde{H}.$$

Since ∂M has a collared neighborhood (see [5]), we can find an embedding

$$\beta\colon (D_2 \cap \partial M) \times [0, 1] \longrightarrow \mathrm{Int}\, \tilde{H}$$

such that $\beta(x, 0) = x$, for all $x \in D_2 \cap \partial M$, and $D_1 \cap \partial M \subset \mathrm{Int}\, \beta((D_2 \cap \partial M) \times [0, 1])$. Choose g sufficiently close to id so that we have

$$\beta((D_2 \cap \partial M) \times [0, 1]) \subset (\psi_1^1 \circ g)^{-1}(\mathrm{Int}\, \tilde{H}).$$

Let $r\colon D_2 \cap \partial M \longrightarrow [0, 1]$ be a map such that $r(x) = 1$, for $x \in D_1 \cap \partial M$, and $r(x) = 0$, for $x \in \mathrm{Bd}_{\partial M}(D_2 \cap \partial M)$. For $0 \le t \le 1$ let

$$N_t = \{\beta(x, s) \mid x \in D_2 \cap \partial M, 0 \le s \le t r(x)\}$$

and define $\gamma_t\colon N_t \setminus \mathrm{Int}\, N_{t/2} \longrightarrow N_t$ to be the unique homeomorphism which linearly stretches fibers over boundary points, i.e. $\gamma_t \circ \beta(x, s) = \beta(x, 2(s - t \cdot r(x)/2))$, for $\beta(x, s) \in N_t \setminus \mathrm{Int}\, N_{t/2}$. Extend γ_t by the identity to a homeomorphism $\delta_t\colon M \setminus \mathrm{Int}\, N_{t/2} \longrightarrow M$. Then define $\psi^2\colon M \times Q_1 \times I \longrightarrow M \times Q_1$ by

$$\psi_t^2 = \begin{cases} (\psi_1^1 \circ g)^{-1} & \text{on } N_{t/2} \times Q_1, \\ (\psi_1^1 \circ g)^{-1} \circ (\delta_t^{-1} \times \mathrm{id}) \circ (\psi_1^1 \circ g) \circ (\delta_t \times \mathrm{id}) & \text{on } (\psi_1^1 \circ g)^{-1}(\tilde{H}) \setminus (N_{t/2} \times Q_1), \\ \mathrm{id} & \text{on } (M \times Q_1) \setminus (\psi_1^1 \circ g)^{-1}(\tilde{H}). \end{cases}$$

Note that ψ^2 is an ambient isotopy such that (1) ψ^2 depends continuously on g, (2) $\psi_0^2 = \mathrm{id}$ and $\psi_1^2 \circ \psi_1^1 \circ g \mid V = \mathrm{id}$, where $V \subset M \times Q_1$ is a neighborhood of $(D_1 \cap \partial M) \times Q_1$, (3) $\psi_t^2 = \mathrm{id}$ for all t if $g = \mathrm{id}$, and (4) $\psi_t^2(M \times B(Q_1)) = M \times B(Q_1)$ and $\psi_t^2 \mid \mathrm{Bd}\, M \times Q_1 = \mathrm{id}$ for all t. Then $\psi = \psi^2 * \psi^1$ clearly fulfills our requirements. For notation we let $g_1 = \psi_1 \circ g$.

2. *Construction of* θ. We will inductively apply Lemma 4.1 to an appropriately chosen handlebody cover of D_1. Since M is PL we can find a sequence

$$X_0 \subset X_1 \subset \cdots \subset X_{n+1} \subset \operatorname{Int} D_2$$

of compact PL submanifolds of M such that X_0 is a neighborhood of $D_1 \cap \partial M$ which lies in V, X_{n+1} is a neighborhood of D_1, and each X_{i+1} is obtained from X_i by adding disjoint handles of index i to X_i. To be more precise, there exist PL embeddings f_j^i: $B^i \times E^{n-i} \longrightarrow \operatorname{Int} D_2$, for $1 \le j \le m_i$, such that the following properties are satisfied:

(1) $f_{j_1}^i(B^i \times E^{n-i}) \cap f_{j_2}^i(B^i \times E^{n-i}) = \varnothing$ for $j_1 \neq j_2$,

(2) $X_{i+1} = X_i \cup (\bigcup_{j=1}^{m_i} f_j^i(B^i \times B^{n-i}))$,

(3) $f_j^i((\operatorname{Int} B^i) \times B^{n-i}) \subset X_{i+1} \setminus X_i$ for all j,

(4) $f_j^i((\operatorname{Bd} B^i) \times B^{n-i}) \subset \partial X_i \setminus \partial M$ for all j.

The construction of the handlebody cover is standard (for example see [12, p. 224]).

Since each $f_j^0(B^0 \times E^n) \times Q_1$ is an open set lying in $\tilde{H}$, we can apply Lemma 4.1 successively to each $f_j^0(B^0 \times E^n) \times Q_1$ (with $K = \varnothing$). Thus for g_1 sufficiently close to id we can construct an ambient isotopy θ^0: $M \times Q_1 \times I \longrightarrow M \times Q_1$ such that (1) θ^0 depends continuously on g_1, $\theta_0^0 = \mathrm{id}$ and $\theta_1^0 \circ g_1 | X_1 \times Q_1 = \mathrm{id}$, (3) $\theta_t^0 = \mathrm{id}$ for all t if $g_1 = \mathrm{id}$, and (4) $\theta_t^0(M \times B(Q_1)) = M \times B(Q_1)$ and $\theta_t^0 | \operatorname{Bd} M \times Q_1 = \mathrm{id}$ for all t.

The next step is to successively apply Lemma 4.1 to each of $f_j^1(B^1 \times E^{n-1}) \times Q_1$ (with $B^1 \times Q_1$ identified with Q and $\operatorname{Bd} B^1 \times Q_1$ identified with K). Thus for g_1 sufficiently close to id we can construct an ambient isotopy θ^1: $M \times Q_1 \times I \longrightarrow M \times Q_1$ such that (1) θ^1 depends continuously on g_1, (2) $\theta_0^1 = \mathrm{id}$ and $\theta_1^1 \circ \theta_1^0 \circ g_1 | X_2 \times Q_1 = \mathrm{id}$, (3) $\theta_t^1 = \mathrm{id}$ for all t if $g_1 = \mathrm{id}$, and (4) $\theta_t^1(M \times B(Q_1)) = M \times B(Q_1)$ and $\theta_t^1 | \operatorname{Bd} M \times Q_1 = \mathrm{id}$ for all t.

In this manner we can inductively construct isotopies $\theta^0, \theta^1, \cdots, \theta^n$ such that $\theta = \theta^n * \cdots * \theta^0$ satisfies (1) θ depends continuously on g_1, (2) $\theta_0 = \mathrm{id}$ and $\theta_1 \circ g_1 | X_{n+1} \times Q_1 = \mathrm{id}$, (3) $\theta_t = \mathrm{id}$ for all t if $g_1 = \mathrm{id}$, and (4) $\theta_t(M \times B(Q_1)) = M \times B(Q_1)$ and $\theta_t | \operatorname{Bd} M \times Q_1 = \mathrm{id}$ for all t.

Thus $\tilde{\phi} = \theta * \psi$ is an ambient isotopy of $M \times Q_1$ which fulfills our requirements.

6. **Proofs of Theorems 1 and 2.** Let X be a compact Q-manifold. Since $H(X)$ is a topological group all we need to do to prove Theorem 1 is show that there exists a neighborhood $\mathfrak{U}$ of id in $H(X)$ and a map ϕ: $\mathfrak{U} \times I \longrightarrow H(X)$ such that $\phi_0(h) = h$, $\phi_1(h) = \mathrm{id}$, and $\phi_t(\mathrm{id}) = \mathrm{id}$ for all $h \in \mathfrak{U}$ and $t \in I$. To construct ϕ we let B be a pseudo-boundary of X and use Lemma 3.4 to get a map ψ: $H(X) \times I \longrightarrow H(X)$ such that $\psi_0(h) = h$ and $\psi_t(h) \in H_B^*(X)$, for all $h \in H(X)$ and $t \in (0, 1]$, and $\psi_t(\mathrm{id}) = \mathrm{id}$ for all t. It follows from Theorem 5.1 that there exists a neighborhood

$\mathfrak{W}$ of id in $H_B^*(X)$ and a map θ: $\mathfrak{W} \times I \longrightarrow H_B^*(X)$ such that $\theta_0(h) = h$, $\theta_1(h) = \text{id}$, and $\theta_t(\text{id}) = \text{id}$ for all $h \in \mathfrak{W}$ and $t \in I$. Choose a neighborhood $\mathfrak{U}$ of id in $H(X)$ and ϵ in $(0, 1)$ so that $\psi_t(h) \in \mathfrak{W}$, for all $h \in \mathfrak{U}$ and $t \in [0, \epsilon]$. Then define ϕ: $\mathfrak{U} \times I \longrightarrow H(X)$ by

$$\phi_t(h) = \begin{cases} \psi_t(h) & \text{for } 0 \leq t \leq \epsilon, \\ \theta(\psi_\epsilon(h), (t - \epsilon)/(1 - \epsilon)) & \text{for } \epsilon \leq t \leq 1. \end{cases}$$

It is clear that ϕ fulfills our requirements.

Before proving Theorem 2 we will consider a special case. Let X be a Q-manifold, $G \subset X$ be open, and let $K \subset G$ be compact. We will show that if h: $G \longrightarrow X$ is an open embedding which is sufficiently close to id, then id_K and $h \mid K$ are ambient isotopic. To see this let B be a pseudo-boundary of X and let $I^*(G, X)$ denote the space of all B-proper open embeddings of G into X. Using Theorem 5.1 there exists a neighborhood $\mathfrak{U}$ of id in $I^*(G, X)$ such that if $g \in \mathfrak{U}$, then id_K and $g \mid K$ are ambient isotopic. Let $I(G, X)$ denote the space of all open embeddings of G into X and let $\mathfrak{W}$ be a neighborhood of id in $I(G, X)$ such that $\mathfrak{W} \cap I^*(G, X) = \mathfrak{U}$. If $h \in \mathfrak{W}$, then Lemma 3.5 implies that there exists a $g \in \mathfrak{U}$ such that $h \mid K$ and $g \mid K$ are ambient isotopic. Since $g \mid K$ is ambient isotopic to id_K it follows that $h \mid K$ is ambient isotopic to id_K, as we wanted.

Passing to the general case we have a Q-manifold X, an open set $U \subset X$, a compact set $C \subset U$, and an isotopy of open embeddings f_t: $U \longrightarrow X$, $t \in I$. Choose any $t_1 \in I$ and let $G = f_{t_1}(U)$, $K = f_{t_1}(C)$. Then the special case treated above implies that if $t \in I$ is sufficiently close to t_1, then $f_t \circ f_{t_1}^{-1} \mid K$ is ambient isotopic to id_K. Thus we can choose a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of I so that for each i, $0 \leq i \leq n - 1$, there exists an ambient isotopy ϕ^i: $X \times I \longrightarrow X$ such that $\phi_0^i = \text{id}$ and $\phi_1^i \mid f_{t_i}(C) = f_{t_{i+1}} \circ f_{t_i}^{-1} \mid f_{t_i}(C)$. Then put $g = \phi^{n-1} * \phi^{n-2} * \cdots * \phi^1 * \phi^0$ and note that $g_0 = \text{id}$, $g_1 \circ f_0 \mid C = f_1 \mid C$.

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